PISOT NUMBERS AND PRIMES

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Abstract

We define and study a transform whose iterates bring to the fore interesting relations between Pisot numbers and primes. Although the relations we describe are general, they take a particular form in the Pisot limit points.

We give three elegant formulae, which permit to locate on the whole semi-line all limit points that are not integer powers of other Pisot numbers.

1. INTRODUCTION AND NOTATIONS

1.1. PRELIMINARY DEFINITIONS

1.1.1. PISOT NUMBERS

A Pisot–Vijayaraghavan number, also called simply a Pisot number, is a real algebraic integer greater than 1 such that all its Galois conjugates are less than 1 in absolute value.

1.1.2. PALINDROMIC, ANTI-PALINDROMIC AND SEMI-PALINDROMIC POLYNOMIALS

A polynomial with real coefficients $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is called palindromic if , for any i, $a_i = a_{n-i}$

A polynomial with real coefficients $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is called anti-palindromic if, for any i, $a_i = -a_{n-i}$

It is called semi-palindromic if, for any even $i < n$, $a_i = -a_{n-i}$ while, for $i=n$ and for any odd i, $a_i = a_{n-i}$

1.1.3. RECIPROCAL, ANTI-RECIPROCAL AND SEMI-RECIPROCAL POLYNOMIALS

A polynomial with real coefficients $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$ is the reciprocal polynomial of $P^*(x) = b_n x^n + b_{n-1}x^{n-1} + \ldots + b_1 x + b_0$ if, for any i, $a_i = b_{n-i}$. The polynomials $P(x)$ and $P^*(x)$ are called each other's anti-reciprocals if, for any i, $a_i = -b_{n-i}$. The polynomials $P(x)$ and $P^*(x)$ are called each other's semi-reciprocals if, for any even $i < n$, $a_i = -b_{n-i}$, while for any odd i $a_i = b_{n-i}$ and if $a_n = b_0$ and if $a_0 = b_n$.

1.2. NOTATIONS

Let $[x]$ designate the *nearest integer function* and let θ be any *Pisot number* which is not a limit point[1] of the closed set of Pisot numbers. To avoid confusion, the brackets [ ] will be used only on this purpose. We shall use brackets $\{\ \}$ to designate only sequences, for example $\left\{\left[\theta^n\right]\right\}_{n\in N}$

We'll use the parenthesis signs ( ) only to indicate the order of elementary arithmetic operations and to fill in the argument of a function or a transform.

Despite some ambiguities, we'll designate by θ_k any Pisot number of degree k (i.e. with minimal polynomial of degree k) that is not a limit point.

We'll drop the index k whenever there is no need of keeping it.

In this article we'll study the iterates of a transform of the sequences $\left\{\theta^n\right\}_{n\in N}$, namely $\left\{\theta^n\right\}_{n\in N} \rightarrow \left\{\theta^n\left(\theta^n - \left[\theta^n\right]\right)\right\}_{n\in N}$.

The sequence $\left\{\theta^n\left(\theta^n - \left[\theta^n\right]\right)\right\}_{n\in N}$ will be considered as the first iterate of the transform, and we shall designate it by $\left\{I\left(\theta^n\right)\right\}_{n\in N}$. Following these notations, $\left\{\left[I\left(\theta^n\right)\right]\right\}_{n\in N}$ is an **integer sequence,** while $\left\{I\left(\theta^n\right)\right\}_{n\in N}$ is, at best, an **almost integer sequence**.)

[1] unless otherwise indicated

$\left\{I^2\left(\theta^n\right)\right\}_{n\in N} = \left\{\theta^n\left(\theta^n\left(\theta^n - \left[\theta^n\right]\right) - \left[\theta^n\left(\theta^n - \left[\theta^n\right]\right)\right]\right)\right\}_{n\in N}$ will be regarded as the second iterate of the same transform.

Generally, if $\left\{I^j\left(\theta^n\right)\right\}_{n\in N}$ is the j - th iterate of this transform, we'll define the (j+1)-*th* iterate of this transform by the formula:

$$\left\{I^{j+1}\left(\theta^n\right)\right\}_{n\in N} = \left\{\theta^n\left(I^j\left(\theta^n\right) - \left[I^j\left(\theta^n\right)\right]\right)\right\}_{n\in N}$$

Of course, $\{\theta^n\}_{n\in \mathbf{N}}$ will be considered as the order 0 iterate of the above-mentioned transform.

For a given θ, we shall sometimes use u_m^k to designate the integer $\left[I^k\left(\theta^m\right)\right]$.

2. THE FIVE MAIN CONJECTURES

2.1. THE PRIME DIVISIBILITY CONJECTURE

For any Pisot number θ_n (which is not a limit point), there are at least two non negative integers k ($k \le n-2$) and $m(\theta_n)$ such as

$$\text{for any prime } p \ge m(\theta_n),\ \left[I^k\left(\theta_n^p\right)\right] \equiv 0 \pmod p$$

2.2. THE CONGRUENCE MODULO PRIME CONJECTURE

For any Pisot number θ_n (which is not a limit point) and for any given non negative integer $k \le n-1$, there is an integer $m(k,\theta_n)$ such as one the four following statements holds:

a) for any prime $p \ge m(k,\theta_n)$ $\left[I^k(\theta_n^p)\right] \equiv 0 \pmod p$

b) for any prime $p \ge m(k,\theta_n)$ $\left[I^k(\theta_n^p)\right] \equiv 1 \pmod p$

c) for any prime $p \ge m(k,\theta_n)$ $\left[I^k(\theta_n^p)\right] \equiv -1 \pmod p$

2.3. THE LINEAR RECURRENCE CONJECTURE

For any Pisot number θ_n (be it a limit point or not) and for any non negative integer k ($k < n-1$), there is an integer $m(k,\theta_n)$ and j signed integers $b_1, b_2,\ldots, b_j$ such as for any $l \geq m(k,\theta_n)$

$$\left[I^k(\theta_n^l)\right] = b_1\left[I^k(\theta_n^{l-1})\right] + b_2\left[I^k(\theta_n^{l-2})\right] + \ldots + b_j\left[I^k(\theta_n^{l-j})\right] = \sum_{i=1}^{j} b_i\left[I^k(\theta_n^{l-i})\right]$$

2.4. THE POLYNOMIAL COEFFICIENTS CONJECTURE

For any Pisot number θ of degree $n \geq 3$ (here θ is supposed to be neither a limit point nor the plastic constant), root of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, there are two integers $j(P)$ and $k(P)$ such as

a) for any $m \geq k(P)$,

$$\left[I^{n-2}(\theta^m)\right] = (-1)^{i(1)}a_1\left[I^{n-2}(\theta^{m-1})\right] + (-1)^{i(2)}a_2\left[I^{n-2}(\theta^{m-2})\right] + \ldots + (-1)^{i(n)}a_n\left[I^{n-2}(\theta^{m-n})\right] = \sum_{j=1}^{n}(-1)^{i(j)}a_j\left[I^{n-2}(\theta^{m-j})\right] \qquad \textbf{(1°)}$$

where i is a function $i:\{a_1, a_2,\ldots, a_n\}\rightarrow\{1, 0\}$

The function i is defined as follows:

If n is odd, then for any k ($1\leq k \leq n$) $i(k) = 0$

If n is even, then for any even k ($1\leq k \leq n$) $i(k) = 0$

and for any odd k ($1\leq k \leq n$) $i(k) = 1$

For any given θ different from the plastic number **(1°)** may be also written as:

$$u_m^{n-2} = (-1)^{i(1)}a_1 u_{m-1}^{n-2} + (-1)^{i(2)}a_2 u_{m-2}^{n-2} + \ldots + (-1)^{i(n)}a_n u_{m-n}^{n-2} = \sum_{j=1}^{n}(-1)^{i(j)}a_j u_{m-j}^{n-2}$$

b) Let again θ be any Pisot number of degree $n \geq 2$ (supposed not to be a limit point), root of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, there is an integer $k_0(P)$ such as for any $m \geq k_0(P)$ the following formula holds:

$$u_m^0 = -a_0 u_{m-n}^0 - a_1 u_{m-(n-1)}^0 - a_2 u_{m-(n-2)}^0 - \ldots - a_{n-1}u_{m-1}^0 = -\sum_{j=0}^{n-1} a_j u_{m-n+j}^0$$

2.5. CHARACTERISTIC POLYNOMIALS

More generally, it seams that, for even k, the characteristic polynomial of the linear recurrence sequence $\left\{\left[I^{\frac{k}{2}+1}\left(\theta_k^n\right)\right]\right\}_{n\geq m(k,\ \theta_k)}$ is anti-palindromic if θ_k is either not a limit point or a limit point of type α (see below, **3.1.,** *a*)). It is semi-palindromic if θ_k is a limit point of type β (see below, **3.1.,** *b*)).

For even or odd k, the characteristic polynomials for the sequences

$$\left\{\left[I^m\left(\theta_k^n\right)\right]\right\}_{n\geq m(k,\ \theta_k)} \text{ and } \left\{\left[I^{k-m+2}\left(\theta_k^n\right)\right]\right\}_{n\geq m(k,\ \theta_k)}$$

are either reciprocal or anti-reciprocal or semi-reciprocal (depending on evenness or oddness of k and m (and of the type of the Pisot number: limit point or not, type of limit point, etc.).

2.6. THE CONSTANT 1 CONJECTURE

For any Pisot number θ of degree $n \geq 3$ (which is not a limit point), root of the polynomial $P(x) = a_n x^n + a_{n-1}x^{n-1} + \ldots + a_1 x + a_0$, there is an integer $j(P)$ such as for any $m \geq j(P)$ $\left|I^{n-1}\left(\theta^m\right)\right| = 1$

3. ON THE PISOT LIMIT POINTS SMALLER THAN 2

3.1. LOGARITHMIC EQUATIONS FOR PISOT LIMIT POINTS

It is well-known that the set of Pisot numbers between 1 and 2 have infinitely many limit points (as well as on the rest of the semi-line). It is also well-known that the sequence of these limit points is made of[2]:

a) the roots α_n of the polynomials $1-x+x^n(2-x)$ $(n \geq 1)$

[2] See also Mohamed Amara, *Ensembles fermés de nombres algébriques,* 1966, *Annales Scientifiques de l'École Normale Supérieure, 3^e^ série, t3, pp. 230-260.* This article has been kindly sent me by Philippe Gille and Jean-Paul Allouche.

b) the roots β_n of the polynomials $\dfrac{1-x^{n+1}(2-x)}{1-x}$ $(n \geq 1)$

c) the number $\hat{o}_2^{'} = 1.90516616775...$, root of the polynomial $1-2x^2-x^3+x^4$ (and, as matter of fact, the square power of the second smallest Pisot number 1.380277…, root of the polynomial $x^4 - x^3 - 1$; it is well-known that if θ is a Pisot number, then for any $m \geq 2$ θ^m is a limit point).

This sequence of limit points verifies the inequalities:

$$\alpha_1 = \beta_1 < \alpha_2 < \beta_2 < \alpha_3 < \hat{o}'_2 < \beta_3 < \ldots < \alpha_n < \beta_n < \alpha_{n+1} < \ldots < 2 \qquad (\blacklozenge)$$

The limit points satisfy the following nice identities:

$$\forall n, \quad \frac{\operatorname{Log}(2-\beta_n)^{-1}}{\operatorname{Log}(\beta_n)} = n+1 \qquad \textbf{(I)}$$

$$\forall n, \quad \frac{\operatorname{Log}(2-\alpha_n)^{-1}}{\operatorname{Log}(\alpha_n)} - \frac{\operatorname{Log}(\alpha_n-1)^{-1}}{\operatorname{Log}(\alpha_n)} = n \qquad \textbf{(II)}$$

Besides, $\dfrac{\operatorname{Log}(2-\alpha_3)^{-1}}{\operatorname{Log}(\alpha_3)} - \dfrac{\operatorname{Log}(\alpha_3-\alpha_1)^{-1}}{\operatorname{Log}(\alpha_3)} = 1$

$$\frac{\operatorname{Log}(2-\alpha_2)^{-1}}{\operatorname{Log}(\alpha_2)} = \frac{5}{2} \qquad\qquad \frac{\operatorname{Log}(\alpha_2-1)^{-1}}{\operatorname{Log}(\alpha_2)} = \frac{1}{2}$$

and

$$\frac{\operatorname{Log}(2-\hat{o}_2^{'})^{-1}}{\operatorname{Log}(\hat{o}_2^{'})} - \frac{\operatorname{Log}(\hat{o}_2^{'}-1)^{-1}}{\operatorname{Log}(\hat{o}_2^{'})} = \frac{7}{2}$$

3.2. CONGRUENCE MODULO PRIME CONJECTURES FOR PISOT LIMIT POINTS SMALLER THAN 2

While reading the statements of this chapter, one has to keep in mind that the Pisot limit points α_n and β_n are Pisot numbers of degree $n+1$.

1°*a*) For any positive integer $n \geq 2$, there is an integer $m(n)$ such as

for any prime $p \geq m(n)$

$$\left[I^0\left(\alpha_n^p\right)\right] \equiv 2 \pmod p$$

1°*b*)

for any prime $p \ge m(n)$ $\left[I^{n-1}(\alpha_n^p)\right] \equiv -1 \pmod p$

1°*c*) For any prime $p \ge m(n)$, for any n and

for any strictly positive $k < n-1$, $\left[I^k(\alpha_n^p)\right] \equiv 0 \pmod p$

1°*d*) For any $n > 0$ and for any integer $l \ge m(n)$

$$\left[I^{2n}\left(\alpha_{2n}^l\right)\right] = 1$$

1°*e*) For any $n > 0$ and for any integer $l \ge m(n)$

$$\left[I^{2n-1}\left(\alpha_{2n-1}^{2l-1}\right)\right] = 1, \text{ while } \left[I^{2n-1}\left(\alpha_{2n-1}^{2l}\right)\right] = -1$$

(note that the left part of the last three equalities might be written without the brackets [ ] of the *nearest integer function.*)

2°*a*) For any positive integer $n > 1$ there is an integer $m(n)$ such as for any prime $p \ge m(n)$ and for any integer $k < n$

$$\left[I^k(\beta_n^p)\right] \equiv 1 \pmod p$$

2°*b*) For any integer $n > 0$ and for any integer $l \ge m(n)$

$$\left[I^{2n}(\beta_{2n}^l)\right] = 1$$

2°*c*) For any integer $n > 0$ and for any integer $l \ge m(n)$

$$\left[I^{2n-1}(\beta_{2n-1}^{2l-1})\right] = 1 \text{ while } \left[I^{2n-1}(\beta_{2n-1}^{2l})\right] = -1$$

(note that the left part of this equality might be written without the brackets [ ] of the *nearest integer function.*)

3°) Remarkably, although it is a limit point (of degree 4), $\hat{o}_2{}'$ behaves in some sense like an ordinary Pisot number:

For any prime $p > 11$, $\left[I^2(\hat{O}_2^{'p})\right] \equiv 0 \pmod p$,

but, on the other hand, for any integer $m \ge 11$

$$\left[I^3(\hat{O}_2^{'m})\right] = -1$$

For sufficiently big l, $[I^{n-1}(\theta'^{l})] = -1$ (where n is the degree of θ') is a typical equality for limit points θ' of the form θ^{m}. (Note again that the left part of this equality might be written without the brackets [ ] of the *nearest integer function.*)

4°) It is very likely that the ***Linear Recurrence Conjecture*** holds as well for all Pisot numbers which happen to be limit points. In other words, for any k smaller than $n - 2$, all sequences $\left\{\left[I^{k}(\theta_n^{l})\right]\right\}_{l\in N}$ are *additive* beginning with some term of rank $m(k,\theta_n)$.

5°) The ***Polynomial Coefficients Conjecture*** holds as well for all Pisot numbers that are also limit points.

5α°) For the numbers α_n the ***Polynomial Coefficients Conjecture*** may be formulated as follows:

For any $n \geq 2$, there is an integer $j(n)$ such as for any $m \geq j(n)$,

$$\left[I^{n-1}(\alpha_n^{m})\right] = \left[I^{n-1}(\alpha_n^{m-(n+1)})\right] + (-2)^{n+1}\left[I^{n-1}(\alpha_n^{m-n})\right] + (-1)^{n}\left[I^{n-1}(\alpha_n^{m-1})\right] \qquad \textbf{(°2)}$$

5β°) For the numbers β_n the ***Polynomial Coefficients Conjecture*** should be formulated as follows:

For any odd $n \geq 2$, there is an integer $j(n)$ such as for any $m \geq j(n)$, we have

$$\left[I^{n-1}(\beta_n^{m})\right] = \left[I^{n-1}(\beta_n^{m-(n+1)})\right] + \sum_{i=1}^{(n+1)/2}\left[I^{n-1}(\beta_n^{m-(2i-1)})\right] - \sum_{i=1}^{(n-1)/2}\left[I^{n-1}(\beta_n^{m-2i})\right]$$

$$= \left[I^{n-1}(\beta_n^{m-(n+1)})\right] + (-1)^{i+1}\sum_{i=1}^{n}\left[I^{n-1}(\beta_n^{m-i})\right] \qquad \textbf{(°3)}$$

while, for any even $n \geq 2$, there is an integer $j(n)$ such as

for any $m \geq j(n)$, we have

$$\left[I^{n-1}(\beta_n^{m})\right] = \left[I^{n-1}(\beta_n^{m-(n+1)})\right] - \left[I^{n-1}(\beta_n^{m-n})\right] - \left[I^{n-1}(\beta_n^{m-(n-1)})\right] -$$

$$\ldots - \left[I^{n-1}(\beta_n^{m-2})\right] - \left[I^{n-1}(\beta_n^{m})\right] = \left[I^{n-1}(\beta_n^{m-(n+1)})\right] - \sum_{j=1}^{n}\left[I^{n-1}(\beta_n^{m-j})\right] \qquad \textbf{(4°)}$$

4. QUADRATIC PISOT NUMBERS

Let θ be a quadratic Pisot number, root of the polynomial $x^2 + a_1x + a_0$.

The sequence $\left\{\left[I\left(\theta^n\right)\right]\right\}_{n\in N}$ is always $\left\{-a_0^n\right\}_{n\in N}$

By the way, 1.5617520677202972947… (which is a root of the 'atypical' polynomial $x^6-2x^5+x^4-x^2+x-1$ and which is not a limit point) is the only known – at least to me – Pisot non quadratic number, such as for any even $l \geq 44$

$$I^{n-1}\left(\theta^l\right) = -1 \qquad \textbf{(*)}$$

while for any odd $l \geq 43$

$$I^{n-1}\left(\theta^l\right) = +1 \qquad \textbf{(**)}$$

5. THE "CONVERGENCE" SPEED CONJECTURE

5.1. How fast sequences generated by the iterates of the I-transform approach integers?

Let θ_n be any Pisot number of degree n (limit point or not). Let $k_1(P)$ be the smallest of the numbers $k(P)$ – respectively $j_1(n)$ the smallest of the numbers $j(n)$ – for which the formulas **(1°)**, **(2°)**, **(3°)** or **(4°)** in the ***Polynomial Coefficients Conjectures*** (see pages 4 and 8) hold true.

For any integer $\lambda \leq n-1$, and for any integers σ and ν

$$k_1(P) \leq \sigma < \nu < \infty \Rightarrow \left|I^{\lambda}(\theta^{\nu}) - \left[I^{\lambda}(\theta^{\nu})\right]\right| < \left|I^{\lambda}(\theta^{\sigma}) - \left[I^{\lambda}(\theta^{\sigma})\right]\right|$$

Following our notations, if θ is a limit point, we'll write:

For any integer $\lambda \leq n-1$, and for any integers σ and ν

$$j_1(n) \leq \sigma < \nu < \infty \Rightarrow \left|I^{\lambda}(\theta^{\nu}) - \left[I^{\lambda}(\theta^{\nu})\right]\right| < \left|I^{\lambda}(\theta^{\sigma}) - \left[I^{\lambda}(\theta^{\sigma})\right]\right|$$

5.2. THE ERRATIC BEHAVIOR CONJECTURE

$k_1(P)$ and $j_1(n)$ may be arbitrarily big. In other words, the sequences of Pisot numbers and the iterates of the transform I give rise to an arbitrarily long initial subsequence with 'erratic' behavior, before they become truly additive sequences.

6. NON INTEGER LIMIT POINTS GREATER THAN 2

The logarithmic equations (I) and (II) can be generalized in the following way (considering integers $m \geq 2$, $n \geq 1$ and $l < m$):

$$\frac{\mathrm{Log}(m-x)^{-1}}{\mathrm{Log}(x)} = n \qquad (\clubsuit)$$

$$\frac{\mathrm{Log}(m-x)^{-1}}{\mathrm{Log}(x)} - \frac{\mathrm{Log}(x-m+l)^{-1}}{\mathrm{Log}(x)} = n \qquad (\heartsuit)$$

Besides, there is an another logarithmic equation, namely

$$\frac{\mathrm{Log}(x-m)^{-1}}{\mathrm{Log}(x)} = n \qquad (\spadesuit)$$

whose real roots – along with the roots of the equations of types (♣) and (♥) – are always Pisot numbers.

Moreover, they always are Pisot *limit points*.

We conjecture that the set of solutions of the three families of equations (♣), (♥) and (♠) contain all Pisot *limit points* that are not integer powers of Pisot numbers.

It is not difficult to establish (♦)-type inequalities (see page 8) for limit points on the semi-line that verify the equations (♣), (♥) and (♠). We shall not do it here, the reader can easily write them by himself. Let us only say that, setting $m = m_0$, all solutions fall in the open[3] interval $]m_0-1, m_0[$.

Let us also point out that, for $m = m_0$, l has m_0-1 possible values. It means

[3] We do not consider here integer limit points : they are trivial.

that in the interval $]m_0-1, m_0[$ we will find m_0-1 infinite sequences of Pisot limit points of the α-type (see page 6, *a*) and (II)). Although the set of Pisot numbers is countable and nowhere dense, one can say, that the set of limit points in the interval $]m_0-1, m_0[$ is 'less dense' than in the interval $]m_0, m_0+1[$ (in the sense that the average interval between two consecutive Pisot limit points that belong to some family of solutions of the families of equations ♣, ♥ or ♠ (indexed on *l*, *m* and *n*) is smaller in $]m_0-1, m_0[$ is 'less dense' than in the interval $]m_0, m_0+1[$.

For the Pisot limit point θ that satisfies the equation

$$\frac{\mathrm{Log}(m_0 - x)^{-1}}{\mathrm{Log}(x)} - \frac{\mathrm{Log}(x - m_0 + l_0)^{-1}}{\mathrm{Log}(x)} = n_0 \qquad (♥_0)$$

we'll state:

1°°) For any positive integer $n \geq 2$, there is an integer $m_0(n)$ such as

$$\text{for any prime } p \geq m_0(n) \ \left[I^{n-2}(\theta^p)\right] \equiv -1 \pmod{p}$$

while,

for any $n \geq 2$ and for any integer $l \geq m_0(n)$

$$I^{n-1}(\theta^l) = 1$$

$$\left[I^0(\theta^p)\right] \equiv m_0 \pmod{p}$$

and, for any prime $p \geq m_0(n)$ and for any strictly positive k smaller than $n - 2$

$$\left[I^k(\theta^p)\right] \equiv 0 \pmod{p}$$

Obviously, **1°°)** happens to be a generalization of **1°*a*)** (see end of page 6).

Conclusion

The methods described in this paper provide an algorithm able, for each given *n*, to generate integer sequences with the following property: For any prime p, $u_p \equiv n \pmod{p}$

This article describes how the iterates of the ***I***-transform generate integer sequences with remarkable properties. Almost nothing is said about the way the iterates of the ***I***-transform generate new recurrence formulae – those of the (conjecturally) linear recurrence sequences that appear at each step of the iterating process.

Examples and results of computations may soon be found in the appendix of the extended version of this paper at http://www.andreivieru.com/page.php?id=4 Interesting examples of fixed points in the fractional part of numbers obtained through the iteration of the ***I*** transform arise in some computations. We believe infinitely many Pisot numbers yield such fixed points (for finitely many integer powers of them). Integer sequences that count subsets (with specific properties) of cyclic graphs appear in some computations.

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APPENDIX:

NOTEBOOK AND NUMERICAL EXAMPLES

1.32471795724474602596090885447809 7…, root of $x(x^2-x-1)+1$ or x^3-x-1

$\{[\theta^n]\}_{n\in N}$ yields the sequence:

1, 2, 2, 3, 4, 5, 7, 9, 13, 17, 22, 29, 39, 51, 68, 90, 119, 158, 209, 277, 367, 486, 644, 853, 1130, 1497, 1983, 2627, 3480, 4610, 6107, 8090, 10717, 14197, 18807, 24914, 33004, 43721, 57918, 76725, 101639, 134643, 178364, 236282, 313007, 414646, 549289, 727653, 963935,… **(I°)**

(Coincides beginning with the 10-th term with A001608 – EOIS i.e. the Perrin sequence)

Beginning from its 13-th term its recurrence additive formula is:

$u_n = u_{n-2} + u_{n-3}$

Characteristic polynomial: x^3-x-1

Beginning with its seventh term any prime p divides u_p, u_{p^2}, etc.

$\{[I([\theta^n])]\}_{n\in N} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in N}$ gives:

0, 0, 1, 0, 0, 2, 1, 5, -5, -6, 1, 6, -12, 13, -7, -5, 18, -25, 20, -2, -23, 43, -45, 22, 21, -66, 88, -67, 1, 87, -154, 155, -68, -86, 241, -309, 223, 18, -327, 550, -532, 205, 345, -877, 1082, -737, -140, 1222, -1959, 1819, -597, -1362, 3181, -3778, 2416, 765, -4543, 6959, -6194, 1651, … **(II°)**

(Coincides beginning with the 10-th term with A078712 – EOIS i.e. the Perrin sequence for negative integers)

For any prime $p > 5$, $u_p \equiv 1 \pmod p$

Beginning with its 13-th term the recurrence relation of sequence **(II°)** is:

$u_n = - u_{n-1} + u_{n-3}$ **(α)**

Characteristic polynomial: x^3+x^2-1

***

1.3802775690976141157673301691822731877816626701558763…

root of $x^2(x^2-x-1)+(x^2-1)$ (i. e. of x^4-x^3-1)

$\{[\theta^n]\}_{n\in N}$ yields the sequence:

1, 2, 3, 4, 5, 7, 10, 13, 18, 25, 35, 48, 66, 91, 126, 174, 240, 331, 456, 630, 870, 1200, 1657, 2287, 3156, 4356, 6013, 8300, 11456, 15812, 21825, 30125, 41581, 57393, 79218, 109343, 150924, 287535, 396878, 547802, 756119, 1043654, 1440532, 1988334, 2744453, 3788107, 5228639,… (beginning from the 22-nd term this sequence coincides with A014097-OEIS) **(III°)**

Recurrence equation (for any $n > 25$):

$u_n = u_{n-1} + u_{n-4}$

Characteristic polynomial: $\boldsymbol{x^4-x^3-1}$

For any prime $p > 19$, $u_p \equiv 1 \pmod{p}$

$\{[\boldsymbol{I}([\theta^n])]\}_{n\in\boldsymbol{N}} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in\boldsymbol{N}}$ yields:

1 , 0 , -1 , -1, 0, -1, -4 , 2, 3, 2, -12, -9, 0, 9, -32, -75, -103, -109, 191, -11, -390, 266, -598, -865, 545, 1718, 201, -2862, -1972, 3514, 5580,-2686,-10865,-2242,17159, 13935, -20202, -36022, 12561, 67917, 20418, -102565, -95976,… **(IV°)**

Recurrence equation[4]:

$\forall n \geq 27 \quad u_n = - u_{n-2} - u_{n-3} + u_{n-4} + u_{n-6}$

characteristic polynomial: $x^6 + x^4 + x^3 - x^2 - 1$

Any prime $p > 19$ divides u_p.

$\{[\boldsymbol{I}^2([\theta^n])]\}_{n\in\boldsymbol{N}} = \{[\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])]\}_{n\in\boldsymbol{N}}$ gives:

2, 0, 0, -1, 0, 3, -3, 4, 6, 12, -11, 16, 13, 30, 18, -78, -102, 39, -228, 54, -406, 77, 92, 127, 150, 169, 219, 277, 319, 388, 496, 596, 707, 884, 1092, 1303, 1591, 1976, 2395, 2894, 3567, 4371, 5289, 6461, 7938, 9660, 11750, 14399, 17598, 21410, 26149, 31997, 39008, 47556, 58146, 71005, 86567, 105705, 129151, 157572,… **(V°)**

(beginning from the 22-nd term this sequence coincides with A050443-OEIS)

Any prime $p > 7$ divides u_p, u_{2p}, u_{5p}, u_{11p}, u_{p^2}, u_{p^3}, etc.

Recurrence equation[5]:

---

[4] Applying backwards the recurrence formula, one can obtain a sequence which is additive from the beginning: (-6,) 0, 2, 3, -6, -5, -1, 14, 2, -15, -23, 22, 39, 0, -82, -32, 98, 136, -109, -266, -11, 479, 266, -598, -865, 545, 1718, 201, … For negative n: -22, -23, 15, 2, -14, -1, 5, -6, -3, 2, 0, -6, (0, 2, 3, -6, -5, -1, 14, 2,…)

[5] Applying backwards the recurrence formula, one can obtain a sequence which is additive from the beginning: 0, 0, 3, 4, 0, 3, 7, 4, 3, 10, 11, 7, 13, 21, 18, 20, 34, 39, 38, 54, 73, 77, 92,

$\forall n \geq 26 \ \ u_n = u_{n-3} + u_{n-4}$ **(β)**

Characteristic polynomial: $\boldsymbol{x^4 - x^1 - 1}$

***

1.443268791270373107628127607386911604676… root of $x^3(x^2-x-1)+(x^2-1)$ i.e. $x^5-x^4-x^3+x^2-1$

$\{[\theta^n]\}_{n\in N}$ gives:

1, 2, 3, 4, 6, 9, 13, 19, 27, 39, 57, 82, 118, 170, 246, 354, 512, 738, 1066, 1538, 2220, 3204, 4624, 6673, 9631, 13900, 20062, 28955, 41790, 60314, 87049, 125635, …, 5250255788 (61-th term), 7577530325 (62-nd term), 10936413033 (63-rd term), **(VI°)**

Recurrence formula:

$\forall n \geq 17 \ \ u_n = u_{n-1} + u_{n-2} - u_{n-3} + u_{n-5}$ **(γ)**

Characteristic polynomial: $x^5 - x^4 - x^3 + x^2 - 1$

For any prime $p \geq 19$ $u_p \equiv 1 \pmod{p}$

$\{[I([\theta^n])]\}_{n\in N} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in N}$ gives:

1, 0, 0, 1, 2, 0, 1, -3, 5, 9, -23, -25, -12, 27, -101, 160, -221, 243, -418, -65, -696, -1268, -1563, 1248, 1951, 5756, 679, -2683, -11947, -12655, 962, 15197, 45181, 19156, -3177, -94489, -105412, -56471, 102951, 327967, 265845, 97216, -645558, -883724, -858593, 525593, 2235697, 2860080, 1670579, -3729403, -7209679, -9195636, 833002, 14184153, 26497384, 20135077, -15973828, -55556927, -85164965, -24683819, 80090805, 220179078, 210901353… **(VII°)**

For any prime $p > 19$, we find $u_p \equiv 1 \pmod{p}$

$\forall n \geq 30$

$u_n = -u_{n-1} + u_{n-2} - 3u_{n-4} - u_{n-5} + 2u_{n-6} + 3u_{n-7} + u_{n-8} - u_{n-9} - u_{n-10}$ **(χ)**

127, 150, 169, 219, 277, 319, 388, 496, 596, 707, 884, 1092, 1303, 1591, 1976, 2395, 2894, 3567, 4371, 5289, 6461, 7938, 9660, 11750, 14399, 17598, 21410, 26149, 31997, 39008, 47556, 58146, 71005, 86567, 105705, 129151, 157572,…is a quite remarkable one. It is a ***generalized Perrin sequence***, and it has a precise combinatorial meaning: its n-th term counts the number of subsets of a n-vertex cyclic graph that skip exactly one or two vertexes.

coefficients :-1, 1, 0, -3, -1, 2, 3, 1, -1, -1,

Characteristic polynomial: $x^{10}+x^9-x^8+3x^6+x^5-2x^4-3x^3-x^2+x+1$

$\{[I^2([\theta^n])]\}_{n\in N} = \{[\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])]\}_{n\in N}$ gives:

-1,0,0, 2, -2, 3, -5, -5, -9, -20, 22, -29, 25, 79, -5, -120, -189, -24, -477, 108, 131, 113, 321, 331, 4, 861, 365, 472, 1043, 1123, 1270, 875, 3635, 1363, 2685, 5940, 3625, 6444, 7207, 13700, 7584, 16033, 27433, 14105, 36274, 40391, 51417, 50523, 84888, 116828, 77066, 193521, 193767, 218163, 313559, 404112, 509699, 478503, 946536, 900867, 1078967, 1727354, 1841315… **(VIII°)**

For any prime $p > 19$, we have $u_p \equiv -1 \pmod p$

$\forall n \geq 30$

$u_n = -u_{n-1} + u_{n-2} + 3u_{n-3} + 2u_{n-4} - u_{n-5} - 3u_{n-6} + u_{n-8} - u_{n-9} - u_{n-10}$ **(δ)**

coefficients: -1, 1, 3, 2, -1, -3, 0, 1, -1, -1

Characteristic polynomial: $x^{10}+x^9-x^8-3x^7-2x^6+x^5+3x^4-x^2+x+1$

One has to note the beautiful symmetry between the coefficients in (χ) (-1, 1, 0, -3, -1, 2, 3, 1, -1, -1) and in (δ) (-1, 1, 3, 2, -1, -3, 0, 1, -1, -1): the two sequences of coefficients both end with -1; the remainders of the two sequences are one another's *cancrizans.*

$\{I^3([\theta^n])\}_{n\in N} = \{[\theta^n(\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])]) - [\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])])]\}_{n\in N}$ gives:

1, 1, 0, 0, -1, -17, -6, 2, 9, 19, -22, -22, 0, 70, 3, 61, -84, -8, -151, -8, -4, -2, -23, 29, -25, 50, -33, 23, -29, -19, 31, -46, 102, -87, 98, -112, 37, -21, -36, 152, -164, 246, -301, 222, -231, 113, 94, -179, 434, -617, 632, -778, 636, -359, 165, 415, -890, 1245, -1829, 1885, -1769, 1579, -580, -366, 1495, -3134, 4020, -4843, 5293, -4234, 2982, -664, -2920… **(IX°)**

The following recurrence formula holds for every $n > 24$:

$u_n = u_{n-2} - u_{n-3} - u_{n-4} + u_{n-5}$ **(ε)**

Characteristic polynomial: $x^5 - x^3 + x^2 + x - 1$

Note the beautiful symmetry between the coefficients of **(γ)** (1, 1, -1, 0, 1) and **(ε)** (0, 1, -1, -1, 1): the two sequences of coefficients both end with 1; the remainders of the two sequences are one another's *cancrizans* with changed signs.

In the sequence **(IX°)** any prime $p > 7$ divides $u_p,\ u_{p^2},\ u_{p^3}$, *etc*.

***

1.46557123187676… root of $x^3(x^2 - x - 1)+(x^2 - 1)$ and of x^3-x^2-1

$\{[\boldsymbol{I}^0([\theta^n])]\}_{n\in\boldsymbol{N}}$

characteristic polynomial x^3-x^2-**1**

$\{[\boldsymbol{I}([\theta^n])]\}_{n\in\boldsymbol{N}} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in\boldsymbol{N}}$ is composed by the follo- wing numbers:

1, 0, 0, -2, -2, -1, -7, 6, 6, -13, 0, 19, -13, -19, 32, 6, -51, 26, 57, -77, -31, 134, -46, -165, 180, 119, -345, 61, 464, -406, -403, 870, -3, -1273, 873, 1270, -2146, -397, 3416, -1749, -3813, 5165, 2064, -8978, 3101, 11042, -12079, -7941, 23121,… **(X°)**

Beginning from the ninth term[6] the above-mentioned sequence is based on the recurrence relation:

$$u_n = -u_{n-2} + u_{n-3} \qquad (\phi)$$

characteristic polynomial: $x^3 + x - 1$

(Beginning with term of rank 5, coincides in absolute values with A112455)

In the sequence **(X°)** any prime $p > 7$ divides $u_p,\ u_{p^2},\ u_{p^3}$, *etc*.

***

1.5015948035390873663777831273710461084863983362535853422483941860633 …

root of $x^4(x^2 - x - 1) + (x^2 - 1)$ (i.e. of $x^6 - x^5 - x^4 + x^2 - 1$)

$[\theta^n]_{n\in\boldsymbol{N}}$ gives:

2, 2, 3, 5, 8, 11, 17, 26, 39, 58, 88, 131, 197, 296, 445, 668, 1003, 1506, 2262, 3397(20), 5100, 7659, 11500, 17269, 25931, 38938, 58469, 87796, 131835, 197962(30), 297259, 446363, 670256, 1006454, 1511285, 2269338, 3407627, 5116874, 7683472, 11537462, 17324592, 26014518, 39063265, 58657195,

[6] 0, 2, -3, -2, 5, -1, -7, 6, 6, -13, 0,… ('corrected' version, obtained applying backwards the linear recurrence formula)

88079340, 132259479, 198600146, 298216948, 447801019, 672415683, 1009695896, 1516154110, 2276649133, 3418604508, 5133358764, 7708224845, 11574630372, 17380404819, 26098325560, 39189110042, 58846163994, 88363094062, 132685562868, 199239951707, 299177676141, 449243643828, 674581921095, 1012948707278, 1521038515100, 2283983530257, 3429617800403, 5149896267210, 7733057473608, 11611918917839, 17436397106144, 26182403287030, 39315360719969, 59035741356370, 88647762443803,… **(XI°)**

The recurrence formula of the sequence **(XI°)** is:

$u_n = u_{n-6} - u_{n-4} + u_{n-2} + u_{n-1}$ (for $n > 37$)

In the sequence **(XI°)**, any prime $p \geq 37$ divides $u_p - 1$

$\{[\boldsymbol{I}([\theta^n])]\}_{n\in \boldsymbol{N}} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in \boldsymbol{N}}$ is composed by the following terms:

-1, 1, 1, 0, -3, 5, 4, -4, -7, 16, -42, 54, 64, 90, -31, 70, 222, 651, 115, -1074, 2259, -1543, 4740, -707, -539, -6867, -9727, 43615, -31203, 78299, 90490, 11969, 269146, -473419, 589516, 654483, -1426645, 1877577, -309207, -4985231, 5828069, -4532303, -8446919, 23965577, -16234282, -4445627, 66474263, -84005038, 9448804, 122105617, -319402615, 171833489, 251827593, -818192094, 1071404169, 290109082, -2090359235, 3644812797, -1800549373, -5339682493, 10831647879, -10979902403, -6872992173, 33851743233, -40374889349, 10312921514, 83560185785, -149245364215, 90784463614, 135432260834, -479598149635, 462203806547, 108987797399,-1193591180555, 1932467768938, -586858855335, -2575886640500, 6175579370019, -5110268302817 **(XII°)**

For any prime $p \geq 37$ we have:

$u_p \equiv 1 \pmod{p}$

$\{[\boldsymbol{I}^2([\theta^n])]\}_{n\in \boldsymbol{N}} = \{ [\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])] \}_{n\in \boldsymbol{N}}$ gives:

0, -1, 1, 2, 2, 4, -6, 2, -10, 22, -42, 8, 92, 87, -133, -321, 290, 358, 418, 1134, 56, 1720, -4967, 6217, 2360, 2812, 23047, -31413, 17516, 1998, 143934, 38080, 61732, 37200, 159785, 196214, 123506, 287132, 408096, 884272, 329312, 1242036, 1525296, 2839064, 1943776, 3640996, 7370822, 7861958, 10662239, 10698464, 30136172, 25938856, 40852612, 46262590, 100305908, 113784297, 125433592, 221508036, 311949520, 482690886, 418290054, 896849720, 1131511592, 1725862912, 1778034086, 3075770278, 4692830884, 5636699800, 7833935976, 10436984277, 18593697828, 19859396006, 30382003152, 40106669780, 66316527980, 79371661424, 106584379717, 164637565870, 227668751470 **(XIII°)**

In the sequence **(XIII°)**, any prime $p \geq 37$ divides u_p.
In **(XIII°)**, any prime $p \geq 7$ divides also u_{3p} and u_p2.

$\{[\boldsymbol{I}^3([\theta^n])]\}_{n\in\boldsymbol{N}} = \{[\theta^n(\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])]) - [\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])])]\}_{n\in\boldsymbol{N}}$ gives:
1, 0, 0, 1, -3, -5, 7, -9, -13, 14, -23, 57, -41, 105, -166, -319, 289, -6, -115, 655, -211, -267, 1747, -5494, 909, 933, -21572, 11979, -1857, -2622, 138973, -2943, 2023, 3873, 13820, 15586, 7473, 2429, -19878, -48495, -42969, -45903, -6365, 83865, 129255, 176361, 182077, -267655, -216364, -447603, -713338, -507415, -23215, 604011, 1809267, 2275482, 1778570, 677668, -2524493, -6030974, -7369167, -7630343, -2077267, 8879489, 18520901, 27963675, 27441725, 5703433, -24271901, -65302142, -101829053, -89352366, -33130613, 74959777, 237838239, 337245137, 325321148, 157229322, -275572067 **(XIV°)**

next iterate - $\{[\boldsymbol{I}^4([\theta^n])]\}_{n\in\boldsymbol{N}}$ - gives, beginning with u_{17}:

…, 306, 20, 19, 698, 28, 46, -1, 5536, 80, 67, 21680, -673, 116, 172, 139128 , 230, 264, 291, 397, 390, 518, 591, 676, 891, 943, 1206, 1376, 1582, 2000, 2210, 2773 …
(XV°)
For any $n \geq 43$, the sequence **(XV°)** is based on the following recurrence formula:

$$u_n = u_{n-6} + u_{n-5} - u_{n-4} + u_{n-2} \qquad \textbf{(ϕ)}$$

In **(XV°)**, any prime $p \geq 29$ divides u_p; any prime $p \geq 7$ divides also u_{3p}.

Next iterate - the fifth - yields, beginning with u_{23}:
…, 47, -5537, 1, -1, -21679, 116, 1, 0, 139129, 270, 1, 1,… **(XVI°)**

ABOUT FRACTIONAL PARTS AS ATTRACTING FIXED POINTS

Let $\theta = 1.3802775690976141157673301691822731877816626701558763\ldots$

Then $\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{n}(\theta^7)\right) = 0.810332335...$

whereas for $n > 1$ $\left[\boldsymbol{I}^{2n}(\theta^7)\right] = 2$ and $\left[\boldsymbol{I}^{2n+1}(\theta^7)\right] = -2$

On the other hand $\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{2n+1}(\theta)\right) = 0.4751114013435952...$

while $\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{2n}(\theta)\right) = 0.6557856100970985...$

This means that there is an attracting orbit of order 2 concerning the fractional part and , taking into account the integer part, an attracting orbit of order 4:

(-0.4751114013435…, -0.65578561009709…, 0. 4751114013435…, 0.65578561009709…)

We also notice that $\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n}(\theta^4)\right) = 0.9051661677540189...$

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n+1}(\theta^4)\right) = 0.344214389902901...$

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n+2}(\theta^4)\right) = 0.24938055765692...$

So we have an attracting orbit of order 3 in the fractional part and of order 6 if we take into account the integer part and the changes of signs.

It happens that the first of these fractional parts is exactly the fractional part of θ^2.

Furthermore $\boldsymbol{I}^4(\theta^3) = 0.4751114013435952...,$ a number we have already seen above

$\boldsymbol{I}^5(\theta^3) = 1.24938055765692...,$ a number whose fractional part we also have already seen above

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n}(\theta^3)\right) = 0.6557856100970986...$ (a number we've also already seen above)

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n+2}(\theta^3)\right) = 0.2493805576569214...$ (a number we've also already seen above)

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{3n+1}(\theta^3)\right) = 0.905166167754018...$ (a number we've also already seen above)

$\lim_{n\to\infty} \text{frac}\left(\boldsymbol{I}^{n}(\theta^5)\right) = 0.2493805576569214...$ (a number we've also already seen above)

Rather surprisingly, frac($\boldsymbol{I}^{\mathbf{4}}([\theta^{20}])$) = frac($\boldsymbol{I}^{\mathbf{5}}([\theta^{20}])$)

frac($\boldsymbol{I}^{\mathbf{4}}([\theta^{24}])$) = frac($\boldsymbol{I}^{\mathbf{5}}([\theta^{24}])$) and

frac($\boldsymbol{I}^{\mathbf{4}}([\theta^{31}])$) = frac($\boldsymbol{I}^{\mathbf{5}}([\theta^{31}])$)

while $[\boldsymbol{I}^4([\theta^{20}])] = [\boldsymbol{I}^5([\theta^{20}])]+1$ and

$[\boldsymbol{I}^4([\theta^{24}])] = [\boldsymbol{I}^5([\theta^{24}])]+1$ and

$[\boldsymbol{I}^4([\theta^{31}])] = [\boldsymbol{I}^5([\theta^{31}])]+1$

***

1.534157744914266915435970076109375701 88… root of $x^4(x^2-x-1)+1$ and of $x^5-x^3-x^2-x-1$

$\{[\boldsymbol{I}^0([\theta^n])]\}_{n\in \boldsymbol{N}} = [\theta^n]_{n\in N}$ gives:

2, 2, 4, 6, 8, 13, 20, 31, 47, 72, 111, 170, 261, 400, 614, 942, 1445, 2216, 3400, 5217, 8003, 12278, 18837, 28899, 44335, 68017, 104350, 160089, 245601, **(XVII°)**

beginning with the 29-th term, any prime p divides u_p.

$\{[\boldsymbol{I}([\theta^n])]\}_{n\in \boldsymbol{N}} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in \boldsymbol{N}}$ yields:

-1, 1, -1, -3, 4, 0, 0, -10, 4, 16, - 21, -1, -52, 44, 102, -267, -373, 1024, 1368, -1285, 2586, 4423, -1103,… **(XVIII°)**

$\{[\boldsymbol{I}^2([\theta^n])]\}_{n\in \boldsymbol{N}} = \{[\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])]\}_{n\in \boldsymbol{N}}$ gives:

0, 0, -1, 2, 2, 7, 1, 12, -13, 29, -33, 71, 66, 127, -302, 195, 290, 353, -1688, 2078, 3850, 1220, 1795 (23-rd term),…, 12877 (29-th term), **(XIX°)**

$\{\boldsymbol{I}^3([\theta^n])\}_{n\in \boldsymbol{N}} = \{[\theta^n(\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])]) - [\theta^n(\theta^n(\theta^n - [\theta^n]) - [\theta^n(\theta^n - [\theta^n])])])]\}_{n\in \boldsymbol{N}}$ gives:

1, -1, -2, 3, 0, -6, 2, 15, 23, 6, 67, 62, -78, 128, -183, -3, -33, -2, 354, -2056, 3788, -32, 70 (23-d term),…, -86 (29-th term),…, 297 (37-th term),…, 1108(41-th term),…, -4229 (47-th term), **(XX°)**

$\{\boldsymbol{I}^4([\theta^n])\}_{n\in \boldsymbol{N}}$ gives:

-1, 0, 1, 3, 2, -6, 3, 7, -7, 1, -32, 63, 79, 129, -25, 1, 1, 1, 345, 2056, 3789, 1, 1,…

(XXI°)

Here is a table of congruence of terms of prime rank in the first iterates of the ***I*-**

transform of the sequence:

$u^0{}_{29}$=245601=29×8469

$u^1{}_{29}$=49765≡1 (mod 29)

$u^2{}_{29}$=12877≡1 (mod 29)

$u^3{}_{29}$ =86 ≡ -1 (mod 29)

$u^4{}_{29}$ =1 (mod 29)

$u^0{}_{41}$= 41751366=41×1018326

$u^1{}_{41}$= 9474076 ≡1 (mod 41)

$u^2{}_{41}$= 685357 ≡1 (mod 41)

$u^3{}_{41}$ = 1108 ≡ 1 (mod 41)

$u^0{}_{31}$=578057=31×8647

$u^1{}_{31}$= -109150 ≡1 (mod 31)

$u^2{}_{31}$=25142≡1 (mod 31)

$u^3{}_{31}$ =63 ≡ 1 (mod 31)

$u^4{}_{31}$ =1 (mod 31)

$u^0{}_{43}$= 98267685 = 43×2285295

$u^1{}_{43}$ = 3292855 ≡1 (mod 43)

$u^2{}_{43}$ = 1284196 ≡1 (mod 43)

$u^0{}_{37}$ = 7536863=37×203699

$u^1{}_{37}$ = 385763 ≡1 (mod 37)

$u^2{}_{37}$ = 181634 ≡1 (mod 37)

$u^3{}_{37}$ = 297 ≡ 1 (mod 37)

We also have the following curious equalities:

frac($\boldsymbol{I}^{\mathbf{3}}([\theta^7])$) = frac ($\boldsymbol{I}^{\mathbf{4}}([\theta^7])$) = frac ($\boldsymbol{I}^{\mathbf{5}}([\theta^7])$) = … etc… **(A)**

frac($\boldsymbol{I}^{\mathbf{3}}([\theta^{14}])$) = frac ($\boldsymbol{I}^{\mathbf{4}}([\theta^{14}])$) = frac ($\boldsymbol{I}^{\mathbf{5}}([\theta^{14}])$) = … etc… **(B)**

frac($\boldsymbol{I}^{\mathbf{3}}([\theta^{20}])$) = frac ($\boldsymbol{I}^{\mathbf{4}}([\theta^{20}])$) = frac ($\boldsymbol{I}^{\mathbf{5}}([\theta^{20}])$) = … etc… **(C)**

frac($\boldsymbol{I}^{\mathbf{3}}([\theta^{21}])$) = frac ($\boldsymbol{I}^{\mathbf{4}}([\theta^{21}])$) = frac ($\boldsymbol{I}^{\mathbf{5}}([\theta^{21}])$) = … etc… **(D)**

That is to say the fractional part is a kind of 'stable fixed point', and the sequences $\{\boldsymbol{I}^{\boldsymbol{n+3}}([\theta^7])\}_{n\in\boldsymbol{N}}$,$\{\boldsymbol{I}^{\boldsymbol{n+3}}([\theta^{14}])\}_{n\in\boldsymbol{N}}$,
$\{\boldsymbol{I}^{\boldsymbol{n+3}}([\theta^{20}])\}_{n\in\boldsymbol{N}}$ and
$\{\boldsymbol{I}^{\boldsymbol{n+3}}([\theta^{21}])\}_{n\in\boldsymbol{N}}$ are kind of stable helixes (mod 1).

Besides, rather curiously,

$\boldsymbol{I}^{\mathbf{4}}([\theta^{10}]) = \boldsymbol{I}^{\mathbf{4}}([\theta^{16}]) = \boldsymbol{I}^{\mathbf{4}}([\theta^{17}]) = \boldsymbol{I}^{\mathbf{4}}([\theta^{18}]) = \boldsymbol{I}^{\mathbf{4}}([\theta^{22}]) = \boldsymbol{I}^{\mathbf{4}}([\theta^{23}]) = 1$

***

1.54521564973275524325255062410511611969147005536 4…   root of $x^5(x^2 - x - 1) + (x^2 - 1)$ i.e. of $x^7 - x^6 - x^5 + x^2 - 1$

$\{[\boldsymbol{I}^0([\theta^n])]\}_{n\in\boldsymbol{N}} = [\theta^n]_{n\in\boldsymbol{N}}$ gives:

2, 2, 4, 6, 9, 14, 21, 33, 50, 78, 120, 185, 286, 442, 684, 1056, 1632, 2522, 3898, 6023, 9306, 14380, 22220, 34335, 53055, 81982, 126679, 195747, 302471, 467383, 722208,…   **(XXII°)**

For any prime p>23, $u_p \equiv 1 \pmod p$

$\{[\boldsymbol{I}([\theta^n])]\}_{n\in\boldsymbol{N}} = \{[\theta^n(\theta^n - [\theta^n])]\}_{n\in\boldsymbol{N}}$ gives:

-1,2, -1, -2, -2, -5,1, -16, 11, -31, -10, 55, 93, 192, -236, 415, 580, 860, -1713, -2581, 1597, 604, 5910, 3528, 7326, -30360,37645, -32791, 20852, 13702, -309596,…   **(XXIII°)**

$\{[\boldsymbol{I}^2([\theta^n])]\}_{n\in\boldsymbol{N}}$ gives:

0, 1, -1, 2, -3, -4, -6, -6, 10, 28, 3, 18, -38, -217, -291, 315, -322, -1026, -436, -661, 1257, 5234, 5935, -11871, 25981, 35922, 32799, 49378, 82302, 137853, 156736,…   **(XXIV°)**

$\{[\boldsymbol{I}^3([\theta^n])]\}_{n\in\boldsymbol{N}}$ gives:

1, 0, -1, -2, -2, 3, 1, 5, -10, 17, -8, -54, 13, 77, 52, -145, 731, 819,1559,-2391,1894,-504,-4279,14629,25506,-40872, -3105, 39999,-580, -126718,9083,…   **(XXV°)**

$\{[\boldsymbol{I}^4([\theta^n])]\}_{n\in\boldsymbol{N}}$ gives:

0, 0, 2, 1, 4, 4, 0, 12, 12, 6, 22, 44, -19, 73, -331, 198, 50, 228, -345, -399, 2085, 56, 4485=23×195, 3774, 25345, -40419, 29, 40774, 608, 127647, 340,…   **(XXVI°)**

$\{[\boldsymbol{I}^5([\theta^n])]\}_{n\in\boldsymbol{N}}$ gives, beginning with n = 17:

..., 50, 50, 1923, 13, 2106, 13, -4462, 498, 25350, -11250, -84, 39505, -17477, 93642, 341,…   **(XXVII°)**

$u^0{}_{37} = 9830976$

$u^1{}_{37} = -2443481 \equiv 1 \pmod{37}$

$u^2{}_{37} = 3650569 \equiv 1 \pmod{37}$

$u^3{}_{37} = 3765676 \equiv 1 \pmod{37}$

$u^0{}_{43} = 133823139$

$u^1{}_{43} = 4072187 \equiv 1 \pmod{43}$

$u^2{}_{43} = 18820283 = 43 \times 437681$

$u^2{}_{53} = 996278895 \equiv 0 \pmod{53}$

$u^3{}_{53} = -14503980 \equiv 0 \pmod{53}$

$u^2{}_{61} = 20711544659 \equiv 0 \pmod{61}$

$u^2{}_{67} = 216526789911 \equiv 0 \pmod{67}$

$u^0{}_{41} = 56047083$

$u^1{}_{41} = 16948253 \equiv 1 \pmod{41}$

$u^2{}_{41} = 8136942 = 41 \times 198462$

$u^3{}_{41} = -215865 = -41 \times 5265$

$u^0{}_{47} = 762935116$

$u^1{}_{47} = -333711609 = -47 \times 7100247$

$u^2{}_{47} = -325550107 \equiv -1 \pmod{47}$

$u^2{}_{59} = 9369332632 \equiv 0 \pmod{59}$

$u^3{}_{59} = 48426787 \equiv 0 \pmod{59}$

***

1.561752067720297294702995364060723780790847286947276642846…

(unknown family) $x^6-2x^5+x^4-x^2+x-1$

2, 2, 4, 6, 9, 15, 23, 35, 55, 86, 135, 211, 329, 514, 802, 1253, 1956, 3055, 4771, 7451, 11637, 18175, 28364, 44329, 69231, 108122, 168860, 263717, 411861, 643225, 1004557, 1568870, 2450185, 3826582, 5976173

$u_n = 2u_{n-1} - u_{n-2} + u_{n-4} - u_{n-5} + u_{n-6}$

Here are terms of prime rank of iteration order 0, 1, 2 and 3:

$u^0{}_{19} = 4771$

$u^1{}_{19} = 1063 \equiv -1 \pmod{19}$

$u^2{}_{19} = 1520 = 19 \times 80$

$u^3{}_{19} = 305 \equiv 1 \pmod{19}$

$u^4{}_{19} = -20$

$u^0{}_{41} = 86715289$

$u^1{}_{41} = -29057357 \equiv -1 \pmod{41}$

$u^2{}_{41} = 5851520 = 41 \times 142720$

$u^3{}_{41} = 424925 \equiv 1 \pmod{41}$

$u^4{}_{41} = 1270 \equiv -1 \pmod{41}$

$u^1{}_{43} = -86762821 \equiv -1 \pmod{43}$

$u^2{}_{43} = 12569760 = 43 \times 292320$

$u^3{}_{43} = 735301 \equiv 1 \pmod{43}$

$u^4{}_{43} = -1721 \equiv -1 \pmod{43}$

$u^0{}_{23} = 28384$

$u^1{}_{23} = 8279 \equiv -1 \pmod{23}$

$u^2{}_{23} = 7728 = 23 \times 336$

$u^3{}_{23} = 1105 \equiv 1 \pmod{23}$

$u^0{}_{47} = 1258253687 \equiv 1 \pmod{47}$

$u^1{}_{47} = 601635343 \equiv -1 \pmod{47}$

$u^2{}_{47} = 97118920 = 47 \times 2066360$

$u^3{}_{47} = 2573063 \equiv 1 \pmod{47}$

$u^4_{23} = -47 \equiv -1 \pmod{23}$ $u^4_{47} = -3244 \equiv -1 \pmod{47}$

$u^0_{31} = 1004557$ $u^0_{53} = 18257476466$

$u^1_{31} = 450181 \equiv -1 \pmod{31}$ $u^1_{53} = 2181080909 \equiv -1 \pmod{53}$

$u^2_{31} = 193688 = 31 \times 6248$ $u^2_{53} = 841810872 = 53 \times 15883224$

$u^3_{31} = 14819$ $u^3_{53} = 18209635 \equiv 1 \pmod{53}$

$u^4_{31} = -218 \equiv -1 \pmod{31}$ $u^4_{53} = 8267 \equiv -1 \pmod{47}$

$u^0_{37} = 14576301$ $u^0_{59} = 264919109897$

$u^1_{37} = -5963515$ $u^1_{59} = -89225694809 \equiv -1 \pmod{59}$

$u^2_{37} = -6393969 \equiv 1 \pmod{37}$, $u^3_{37} = -2091239 \equiv 1 \pmod{37}$,

$u^4_{37} = -126912 \equiv -2 \pmod{37}$,

The numbers indicated in parenthesis after each integer represents its rank in some sequence generated by the ***I***- transform iterations:

-1721 (43), 179 (44), 2422 (45), -1 (46), 3244 (47), -204 (48), 4507 (49), 719 (50), -6050 (51), -1362 (52), 8267 (53), 2624 (54), -10996 (55), -4371 (56)

-1721+2*179+2422+3244+204=4507

179+2*2422-1+204-4507=719

2422-2*1-3244-4507-719= -6050

-1-2*3244-204-719+6050= -1362

-3244-2*204+4507+6050+1362==8267

-204+2*4507+719+1362-8267=2624

4507+2*719-6050-8267-2624= -10996

719-2*6050-1362-2624+10996= -4371

$u_n = u_{n-6} + 2u_{n-5} + u_{n-4} + u_{n-2} - u_{n-1}$

$u_n = u_{n-6} + 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = u_{n-6} + 2u_{n-5} - u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = u_{n-6} - 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = -u_{n-6} + 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = u_{n-6} + 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = u_{n-6} + 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = u_{n-6} + 2u_{n-5} + u_{n-4} - u_{n-2} - u_{n-1}$

$u_n = 2u_{n-1} - u_{n-2} + u_{n-4} - u_{n-5} + u_{n-6}$

***

2.05596739671281870311852547611254… root of x^5 - $2x^4$ - 1

$\{[\theta^n(\theta^n - [\theta^n])]\}_{n\in N}$ is composed by the following numbers:

0, 1, -3, -2, -10, -36, 43, 80, 240, 630, 1276, 1136, -4134,
-8176, -10474,…

$\{[\boldsymbol{I}^2([\theta^n])]\}_{n\in N}$ gives, beginning with $n = 10$:

…, 74, 176, 408, 832, 508, 1450, …

Prime p divides term of rank p.

$\{[\boldsymbol{I}^3([\theta^n])]\}_{n\in N}$ gives, beginning with $n = 10$, gives:

…, -5, 0, 32, -52, 28, -5, -64, 136,…

recurrence relation:

$u_n = u_{n-5} - 2u_{n-4}$ **(ε)**

***

2.029074361329942335954938538737600005337923126369… root of x^6 **- $2x^5$ - 1**

Here is $\{[\boldsymbol{I}^3([\theta^n])]\}_{n\in N}$ beginning with $n = 19$:

-1520, -2320, 9405, -5632, -5152, 14321, 4640, -56160,…

$\{[\boldsymbol{I}^4([\theta^n])]\}_{n\in N}$ beginning with $n = 19$ gives:

…, 0, 80, 168, 132, 46, 6, 160, 416, 432, 224, 58, 326, 992, 1280, 880, 340, 710,
2310, 3552, 3040, 1560, 1760, 5330, 9414, 9632, 6160,…

where any prime $p \geq 19$ divides u_p; any prime $p \geq 11$ divides u_{2p} and u_{3p} ; any prime $p \geq 7$ divides u_{4p}, etc.

Recurrence formula (for $n \geq 25$):

$u_n = u_{n-6} + 2u_{n-5}$

Ln(3.10380340273553653316494733282893-

3)/Ln(3.10380340273553653316494733282893)=

-1.99999999999999999999999999999998268977605172278

Ln(3.03574411229409646-3)/Ln(3.03574411229409646)=

-3.0000000000000151761926231155911112001726849712

Ln(2.10691934037621-2)/Ln(2.10691934037621)=

-3.000000000001043664332341354210165261954345994 7

Ln(2.02907436132994233595493853873 76-

2)/Ln(2.0290743613299423359549385387376)=

-5.00000000000000000000000000000027805932115863

Ln(2.205569430400589 -2)/Ln(2.205569430400589)=

-2.00000000000009570675490711241250210636641748 8

LIMIT POINTS SMALLER THAN 2

1.6180339887498948482045868343656381177203091798 1…
$x^2 - x - 1$ ($\beta_1=\alpha_1$)
$I^0(\theta^{41}) \equiv 1$ (mod 41)
$u_n = u_{n-1} + u_{n-2}$
$\{I(\theta^n)\}$: -1, -1, 1, -1, 1,

1.83928675521416113255185256465328660 02…
$x^3 - x^2 - x - 1$ (β_2)
$I^0(\theta^{41}) \equiv 1$ (mod 41)
$I(\theta^{37})$, $I(\theta^{38})$, $I(\theta^{39})$, $I(\theta^{40})$, $I(\theta^{41})$,…
- 62751≡ 1(mod 37), -112555, 289167, -239363, -162359 ≡ 1, …
$u_n = - u_{n-1} - u_{n-2} + u_{n-3}$
characteristic polynomial: $x^3 + x^2 + x - 1$
(see A075298-OEIS "inverted tribonacci numbers")
-112555 - 289167 + 239363 = -162359
$\boldsymbol{I}^2(\theta^{37})$=1, $\boldsymbol{I}^2(\theta^{36})$=1 $\boldsymbol{I}^2(\theta^{35})$=1 $\boldsymbol{I}^2(\theta^{34})$=1

1.9275619754829253042619058617366221294183538643 5…
$x^4 - x^3 - x^2 - x - 1$ (β_3)
$I^0(\theta^{41}) \equiv 1$ (mod 41)

2, 4, 7, 14, 27, 51, 99,191, 367, 708, 1365, 2631, 5071, 9775, 18842, 36319, 70007, 134943, 260111, 501380, 966441, 1862875,…

$I^1(\theta^{31}) \equiv 1 \pmod{31}$

0, -1, 1, 3, -10, 15, -13, -81, 127, 58, -175, -329, 885, 31, -1424, -833, 5543, -2181, -9233, 2298, 31025, -27893, -49495, 54879, 150416, -245697, -204965, 526887, 570895, -1801670, -407711, 3882303, 946397, -11542929, 3442672, 24121039, -10317745,…

(beginning with the sixth term coincides in absolute values with A074193-OEIS and A074453-OEIS; see also A074058-EOIS, A073817)

for $n > 11$

$u_n = -u_{n-1} - 2u_{n-2} - 2u_{n-3} + 2u_{n-4} - u_{n-5} + u_{n-6}$

coefficients 1 -1 2 -2 -2 -1

characteristic polynomial:

$$x^6 + x^5 + 2x^4 + 2x^3 - 2x^2 + x - 1$$

I^2:

0, 0, 1, 4, -10, -1, 1, 15, 19, 4, 1, 31, 53, 27, 6, 63, 137, 107, 39, 132,…, 2481 ≡ 1 (mod 31), 2303, 4621, 7683, 7846, 7087, 11545≡ 1 (mod 37), … (see A073937 and A074058)

I^3:

-1, -1 (frac), 0.99999999, 3.23428523300352175366178110415 (frac), -5, -1.00000000…, 1(7), -1(8), 1, -1, 1, -1, 1, -1, 1, -1, 1, -1,

$x^4 - x^3 - x^2 - x - 1$ (β_3)

$u_n = u_{n-1} - u_{n-2} + u_{n-3} + u_{n-4}$

characteristic polynomial: $x^4 - x^3 + x^2 - x - 1$

1.965948236645485337189937375934401396151327177 45…

$x^5 - x^4 - x^3 - x^2 - x - 1$ (β_4)

$\boldsymbol{I^0}$:

2, 4, 8, 15, 29, 58, 114, 223, 439, 862, 1695, 3333, 6553, 12883, 25327, 49791, 97887, 192441, 378329, 743775, 1462223, 2874655, 5651423, 11110405, 21842481, 42941187,… (see A074048 : "These pentanacci numbers follow the same pattern as Lucas, generalized tribonacci (A001644) and generalized tetranacci (A073817) numbers: Binet's formula is a(n)=r1^n+r2^n+r3^n+r4^n+r5^n, with r1, r2, r3, r4, r5 roots of the characteristic polynomial. a(n) is also the trace of A^n, where A is the pentamatrix ((1,1,0,0,0),(1,0,1,0,0),(1,0,0,1,0),(1,0,0,0,1),(1,0,0,0,0)).")

$\boldsymbol{I^1}$:

0, -1, -3, -1, 11, -15, -56, 31, -140, 366, 815, 758, -311, -3021, -3796, 13759, 7039, -16086, -45295, -3681, 204684, 10431, 365377, -507914, 618001, 2642435, -1427468, -6214881, -3341553, 16185322, 27959273, -42625665, -85186108, 23867663, 286766767, 193092086, -854985639, -900760205, 1376448220, 4034073631, -694664065, -13879346838, -5210320319, 31518148287, 44458083740, -58099066369, -188333702913,… (see A123127 beginning with the 8-th term: "Also sum of the successive powers of all combinations of products of two different roots of the quintic pentanacci polynomial X^5-X^4-X^3-X^2-X-1; namely (X1 X2)^n + (X1 X3)^n + (X1 X4)^n + (X1 X5)^n + (X2 X3)^n+ (X2 X4)^n + (X2 X5)^n + (X3 X4)^n + (X3 X5)^n + (X4 X5)^n, where X1,X2,X3,X4,X5 are the roots. A074048 are the coefficients, with changed signs, of X^4 in the characteristic polynomials of the successive powers of the pentanacci matrix or

(X1)^n+(X2)^n+(X3)^n+(X4)^n+(X5)^n. … a(5)=49 because characteristic polynomial of fifth power of pentanacci matrix M^5 is X^5-31X^4+49X^3-31X^2+9X-1 in which coefficent of X^3 is 49)

$u_n = -u_{n-10} + u_{n-9} + u_{n-7} - u_{n-6} + 6u_{n-5} - 3u_{n-4} - 3u_{n-3} - 2u_{n-2} - u_{n-1}$
coefficients: -1 1 0 1 -1 6 -3 -3 -2 -1
characteristic polynomial:
$x^{10} + x^9 + 2x^8 + 3x^7 + 3x^6 - 6x^5 + x^4 - x^3 - x + 1$

$\boldsymbol{I}^2$:
0, 2, 0, 1, -6, -20, -56, 33, 4, -356, 199, 10, 209, 113, 604, 1473, 375, 1174, 1521, 2721, 9580, 5501, 6671, 14346, 15681, 57409, 56596, 44577, 112463, 119382, 333313, 480641, 360628, 800973, 1007191, 1988362, 3628369, 3160689, 5525420, 8309793, 12812583, 25517654, 27239985, 38743233, 64996444, 90150525, 173552671,… (see A123126 Also sum of successive powers of all combinations of product of three different roots of quintic pentanacci polynomial X^5-X^4-X^3-X^2-X-1 Let roots are X1,X2,X3,X4,X5 (X1 X2 X3)^n + (X1 X2 X4)^n + (X1 X2 X5)^n + ... + (X3 X4 X5)^n.)

$u_n = -u_{n-10} - u_{n-9} - 2u_{n-8} - 3u_{n-7} - 3u_{n-6} + 6u_{n-5} - u_{n-4} + u_{n-3} + u_{n-1}$
Coefficients (verified for n>=21 , starting term = 9580): -1 -1 -2 -3 -3 6 -1 1 0 1
characteristic polynomial
$x^{10} - x^9 - x^7 + x^6 - 6x^5 + 3x^4 + 3x^3 + 2x^2 + x + 1$

$\boldsymbol{I}^3$:
0, -1, -3, 1, -14, -15, 0, 1, 1, 122, 23, -5, 1, 1, -39, 65, -33, 7, 1, -79, 169, -131, 47, -5, -159, 417, -431, 225, -57 (see also A074062 beginning with the 11-th term),…, -1279 (32-nd term), 881, -339, -569, 2299, -3551 ≡1 (mod 37),…
$u_n = -u_{n-1} - u_{n-2} - u_{n-3} - u_{n-4} + u_{n-5}$
characteristic polynomial: $x^5 + x^4 + x^3 + x^2 + x - 1$

$\boldsymbol{I}^4$:
1, 2, 0, 1, -3, -6, 0, 1, 1, 1, -18, 1, 1, 1, 1, 1,…

1.98358284342432633038562929339142575273008086556 9…
$x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ (β_5)
8 (twenty-first term), -1 (22), 1 (23), 95 (24), 201 (25), 155 (26), 55 (27), 6 (28), 1 (29),…
$u_n = u_{n-1} - u_{n-2} + u_{n-3} - u_{n-4} + u_{n-5} + u_{n-6}$
$\boldsymbol{I}^5(\theta^{21})=1$, $\boldsymbol{I}^5(\theta^{20})=-1$

1.99196419660503502109774175458437496347931896005315799524478215…
$x^7 - x^6 - x^5 - x^4 - x^3 - x^2 - x - 1$ (β_6)

-55 (twenty-first term), -2992 (22), -45 (23), 9(24), 1 (25), 1 (26), 1 (27), -111 (28), 233 (29), -179 (30), 63 (31), -7 (32),…
$u_n = -u_{n-1} - u_{n-2} - u_{n-3} - u_{n-4} - u_{n-5} - u_{n-6} + u_{n-7}$

-22 (fourteenth term), -31 (15), 599 (16), 1387 (19), 29 (20), 69 (21), 120 (22), 163 (24), 149 (25), 90 (26),…
$-x^7 + 2x^6 - x + 1$ (α_6)
$u_n = u_{n-1} - 2u_{n-6} + u_{n-7}$

1.9671682128139660358552623604855622911088040495998…
-316, 365, -316, 120, $I^4(\theta^{29})$ = 289, -670, 1084, -1351,
$-x^6 + 2x^5 - x + 1$ (α_5)
$u_n = - u_{n-1} + 2u_{n-5} + u_{n-6}$

1.**9331849818995204467914240303356315863751837844798**…
$I^0(\theta^{37}) = 39094554388 \equiv 2 \pmod{37}$
$I^1(\theta^{37}) = 676223988 \equiv 0 \pmod{37}$
$I^2(\theta^{37}) = 12105808 \equiv 0 \pmod{37}$
$I^3(\theta^{37}) = -75 \equiv -1 \pmod{37}$,
$-x^5 + 2x^4 - x + 1$ (α_4)
$u_n = u_{n-1} - 2u_{n-4} + u_{n-5}$

I^0:
2, 4, 7, 14, 27, 52, 101, 195, 377, 729, 1409,…

I^1:
0, -1, 2, 0, 0, 10, -10, 13, 39, 8, 441, 1262, -572, -1269, 8829, -9622, 4862, 21079, -76076, 78636, -23666, -254630, 647174, -650138, -59950, 2685564, -5530560, 5170919, 3276130, -26460385, 47448786, -38092342, -53419922, 251479128, -404827199, 247939355, 676223988, -2340149024, 3396204169, -1262328972, -7630831604, 21438716022, -27728243840, 2121123778, 80695406034, -193463732998, 218119288992, 65036507846,…

Coefficients (verified for n>=22, starting term = -254630): -1 0 2 1 3 -2 -7 2 1 0

I^2:
0, 0, -3, -6, 0, 13, 44, 73, 30, 54, 683, 173, 286, 333, 1308, 716, 1870, 2434, 6270, 4630, 10706, 16856, 29348, 32853, 59950,106058, 145803, 217545, 344520, 620118, 796576, 1339276, 2033837, 3488438, 4665745, 7895090, 12105808, 19474358, 28020814, 45776070, 71597152, 109999434, 167765532, 265180364, 419012895, 632118664, 992790964, 1542760229,…

Coefficients (verified for n>=22 , starting term = 16856): -1 0 1 2 -7 -2 3 1 2 0

1.8667603991738620929908720624947194…
$I^0(\theta^{31}) = 253364614 \equiv 2 \pmod{31}$
$I^1(\theta^{31}) = 3180600 \equiv 0 \pmod{31}$
$I^2(\theta^{29}) = -1886 \equiv -1 \pmod{29}$, 772, $2107 \equiv -1 \pmod{31}$, -4307, 3965
$-x^4 + 2x^3 - x + 1$ (α_3) $x^4 - 2x^3 + x - 1$ (2, 0, -1, 1)
$u_n = -u_{n-1} + 2u_{n-3} + u_{n-4}$ (-1, 0, 2, 1)

I^0:
2, 3, 7, 12, 23, 42, 79, 147, 275, 514, 959, 1791, 3343, 6241,…

I^1:
0, 2, -3, 2, -7, 13, 0, 70, 81, -47, 330, -241, 390, -1262, 864, -2238, 4590, -3809, 10830, -16603, 18531, -47522, 62100, -90321, 196965, -245702, 424359, -793798, 1026600, -1911494, 3180600, -4443198, 8312841, -12867302, 19488435, -35274121, 52958100, -85315322, 147604869, -221782803,…
for $n > 14$
$u_n = u_{n-2} - 3u_{n-3} - u_{n-4} + u_{n-6}$
coefficients: 1 0 -1 -3 1 0
characterisitic polynomial:
$x^6 - x^4 + 3x^3 + x^2 - 1$
Coeffs (verified for n>=16 , starting term = -2238): -3 1 3 8 -6 1 3
Coeffs (verified for n>=16 , starting term = -2238): -2 1 2 5 -5 1 2
Coeffs (verified for n>=16 , starting term = -2238): -1 1 1 2 -4 1 1
Coeffs (verified for n>=15 , starting term = 864): 0 1 0 -1 -3 1 0
Coeffs (verified for n>=16 , starting term = -2238): 1 1 -1 -4 -2 1 -1
Coeffs (verified for n>=16 , starting term = -2238): 2 1 -2 -7 -1 1 -2
Coeffs (verified for n>=16 , starting term = -2238): 3 1 -3 -10 0 1 -3

I^2:
0, -1, -1, -3, -11, 20, -8, -65, 5, -14, 32, -9, -14, 64, -50,…

I^3:
-1, 0, -3 (frac), -1, -3, 19 (frac), 1, 12, 1, -1, 1, -1, 1,…

For θ = 1.75487766624669276004950889635827....
$I^0(\theta^{37})$ =1089264462≡2 (mod 37), $I^0(\theta^{38})$ = 1911525877, $I^0(\theta^{39})$ = 3354494070, $I^0(\theta^{40})$ =5886726725, $I^0(\theta^{41})$ ≡ 2 (mod 41),…
$x^3 -2x + x - 1$
(see A109377 : A coin is tossed n times and the resultant strings of H's and T's are arranged in a circular (cyclic) manner (i.e. the outcome of the n-th toss is chained to the outcome of the first toss). Then the above sequence represents the number of strings, out of total possible strings of n tosses (n>1), having no isolated H, (by an isolated H, we mean single 'H' which is preceded and succeeded by a 'T'), when the resultant strings are arranged and studied in circular manner. Illustration: In the following string of 10 tosses, 'HHTHTHTTTH', there are only 2 isolated H's, namely the H's at toss number 4 and 6. whereas in the string 'THTHTHTTTH', there will be 4 isolated H's, namely at toss number 2,4,6 and 10. In the string 'HHTTHHHTTH' there is no isolated H, as the H at the 10th toss when chained to the first toss, will no longer be the isolated H, but a triple H.

If a(k) denotes the k-th term(k>4), of the above sequence then a(k)=2a(k-1)-a(k-2)+a(k-3), with a(2)=2, a(3)=5, a(4)=10. Also the k-th term, a(k)(k>5), of this sequence, can be obtained by the formula, a(k)=a(k-1)+a(k-2)+a(k-4), (previous 4 terms are needed), where a(2)=2, a(3)=5, a(4)=10, a(5)=17.
a(n) = P(2*n + 4) where P is the Perrin sequence (A001608). a(n) is asymptotic to r^(n+2) where r is the real root of x^3 -2*x^2 +x -1 (A109134). For n>2, a(n) =

round(r^(n+2)). - Gerald McGarvey (gerald.mcgarvey(AT)comcast.net), Jan 12 2008)
$I^1(\theta^{37})$ = 16279 ≡ -1(mod 37), 87118, 8907, -149050, $I^1(\theta^{41})$ =
-79746 ≡ -1 (mod 41),…, -4181701 ≡ -1 (mod 53),
I^0:
2, 3, 5, 9, 17, 29, 51, 90, 158,…

I^2:
0, 0, 2, 5, -6, 6, 13, -5, -25,…

I^3:
1, 1, 1, -4, 1, 1, 1, 1, 1,…

$-x^3 + 2x^2 - x + 1$ (α_2)
$u_n = u_{n-1} - 2u_{n-2} + u_{n-3}$

1.9051661677540189095727878303640157935069696493O3…
$I^0(\theta^{41})$= 300087072040 ≡ 1 (mod 41) 22778013372(37), 43395900445 (38), 82676401347(39), 157512282718(40)
A014097
$I^1(\theta^{31})$ ≡ 2 (mod 31)
$I^2(\theta^{37})$ =2571685 = 37*69505, 3832532, 5710903, 8511070, 12682653 = 41*309333,
$I^3(\theta^{11})= I^3(\theta^{12})= I^3(\theta^{13})= I^3(\theta^{14})=$ -1
$x^4 - x^3 - 2x^2 + 1$ (-1, 0, 2, 1)
$u_n = 2u_{n-2} + u_{n-3} - u_{n-4}$ (-1, 1, 2, 0)

I^1:

0, -1, -1, 2, 2, -9, 9, -75, -109, -11, 266, -865, 1718, -2862, 3514, -2686, -2242, 13935, -36022, 67917, -102565, 114042, -52714, -167329, 649777, -1479198, 2593295, -3592846, 3420146, -78086, -9485998, 28694402, -59252689, 96444258, -121419167, 90763631, 72119366, -474315702, 1219513931, -2315589027,…

Coefficients (verified for n>=15 , starting term = 3514): -1 -2 1 5 1 -2